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The ultraviolet problem and analytical properties of classical field theories

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Abstract. The absence of ergodicity is investigated analytically and numerically for classical field theories and for Euler equations in two dimensions. In this case we extend the arguments of Patrascioiu to inviscid two-dimensional fluid dynamics. We comment on the risks of truncation introduced by a numerical simulation of continuous systems reflecting on the analytical properties of the solution of the field equations.

It appears that ergodicity is a property only of the discretised problem.

Our considerations are tested on a simple model of a radiant cavity, which shows the absence of the ultraviolet catastrophe and the possibility of wrong interpretations of numerical simulations of field theories.

1. Introduction

The ergodic hypothesis, i.e. the equivalence between time and ensemble averages, is crucial in establishing a relation between Boltzmann's dynamical approach and Gibb's statistical mechanics.

For Hamiltonian systems with a finite (even if large, $\sim 10^3$) number of degrees of freedom (N) it has been ascertained that at low energy density one does not meet such useful properties as ergodicity or equipartition of energy among the degrees of freedom. Some fundamental mathematical theorems (KAM theorem above all, see Kolmogorov (1954), Arnold (1963) and Moser (1962)) and a host of numerical investigations (see, for example, Fermi *et al* (1955), Izrailev and Chirikov (1966), Bocchieri *et al* (1970), Callegari *et al* (1979), Benettin *et al* (1980) and Livi *et al* (1985)) depict a reality much more similar to linear or integrable systems with ordered motions, regular orbits and periodic behaviour.

The extension of these results to continuous systems (i.e. classical fields) is very difficult both from the mathematical and from the numerical points of view. In the last case the results are often unreliable or difficult to interpret.

It is not useless to underline that the thermodynamic limit ($V \to \infty$, $N \sim V$; where V is the volume) does not coincide with the continuum limit ($V = \text{constant}$, $N \to \infty$): therefore the numerical study of the problem of equipartition of energy is somewhat

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different in these two limits, as we shall show in the following. The continuum limit has been recently studied by Patrascioiu (1984) who has shown that time averages and microcanonical averages do not coincide for a large class of classical field equations, derived from Hamiltonians, whose global solutions have some regularity properties.

We are going to show that such conclusions can be extended to the equations describing the motion of a perfect fluid in two dimensions (Euler equations). In particular, following simple dynamical arguments (i.e. using some properties of the evolution equations), we can derive that a small fraction of energy is flowing towards high wavenumber modes, thus removing the ultraviolet (UV) catastrophe, which follows from classical statistical mechanics. What is peculiar is that one is led to the conclusion that the UV catastrophe is incompatible with the regularity of the solution of field equations.

In § 2 we shall outline the results obtained by Patrascioiu in order to facilitate the reader. In § 3 we shall discuss the two-dimensional Euler equations and in § 4 we will comment in general on the problem of the analyticity of the solution of a field equation in connection with numerical simulations, and we shall underline that in some cases the apparent UV catastrophe is a property of the truncated equation (studied numerically) but not of the true continuous system. In § 5 we shall present a numerical simulation on a simple model of a radiant cavity in order to clarify the arguments discussed in § 4. Section 6 is reserved for some conclusions.

2. Failure of ergodicity in classical field theories

In this section we shall repeat an argument of Patrascioiu (1984) on the absence of ergodicity in some classical field theories.

Let us consider a d-dimensional field theory whose Lagrangian density is

$$\mathscr{L}=\tfrac{1}{2}[\dot{\varphi}^2-(\nabla\varphi)^2-m^2\varphi^2]-V(\varphi). \tag{2.1}$$

The corresponding equation of motion is

$$\partial_t^2\varphi=(\Delta-m^2)\varphi-\partial V/\partial\varphi. \tag{2.2}$$

Let us consider a toroidal domain $[0, L]^d$ (but the argument is valid for any bounded domain). In this case one can expand $\varphi(x, t)$ in terms of the eigenfunctions $u_n(x)=\exp(\mathrm{i}\boldsymbol{k}_n\boldsymbol{x})$, $(\boldsymbol{k}_n=2\pi\boldsymbol{n}/L)$, of the operator $-\Delta+m^2$:

$$\varphi(x, t)=\sum_n a_n(t)u_n(x). \tag{2.3}$$

If $V(\varphi)\propto|\varphi|^p$ then in Fourier space

$$V(\{a_n\})\propto\sum_{n_1,\ldots,n_{p-1}} a_{n_1}\ldots a_{n_{(p-1)}}a_{(n_p-n_1-\ldots-n_{(p-1)})}. \tag{2.4}$$

Now we are ready for the argument: (i) if the initial condition is generic and it has a finite energy; (ii) if a global smooth solution exists at any time; being the Hamiltonian the sum of positive-definite terms following Patrascioiu (1984), with the aid of pure dimensional analysis, one can derive in the asymptotic region ($|\boldsymbol{n}|\gg1$) the following bounds:

$$\sum_n|\dot{a}_n|^2<\infty\Rightarrow|\dot{a}_n|<\text{constant}|\boldsymbol{n}|^{-d/2-\varepsilon} \tag{2.5a}$$

$$\sum_n|\boldsymbol{n}|^2|a_n|^2<\infty\Rightarrow|a_n|<\text{constant}|\boldsymbol{n}|^{-1-d/2-\varepsilon} \tag{2.5b}$$

$$V(\{a_n\})<\infty\Rightarrow|a_n|<\text{constant}|\boldsymbol{n}|^{(1-p)d/p-\varepsilon}. \tag{2.5c}$$

For $d \leqslant 2$, the cases in which we are interested in this paper, the second bound is stronger than the third for any value of p. It results that the linear term of the equation of motion is much greater than the non-linear one for $|\boldsymbol{n}| \gg 1$. This means that for large values of $\boldsymbol{n}$ the motion is practically harmonic and therefore the time average of $\dot{a}_n^2(t)$: $\overline{\dot{a}_n^2(t)}$ does not differ very much from the value it would have in the linear case ($V=0$). In other words, if the energy is initially fed in the lower modes only a very small fraction reaches the higher modes. As claimed in the introduction this effect prevents the UV catastrophe. On the contrary the Gibb's microcanonical average of $\dot{a}_n^2$ is independent of $\boldsymbol{n}$.

The generalisation of the argument of Patrascioiu to other forms of the potential (i.e. $V(\varphi) \propto |\nabla\varphi|^p$) is straightforward and again one obtains that equipartition of the energy among the normal modes of the corresponding linear problem is not reached.

3. Absence of equipartition in two-dimensional Euler equations

The situation that we have described in the previous section for Lagrangian field theories is also present in the study of the dynamics of an inviscid incompressible fluid in two dimensions. Euler equations describe the evolution of the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$:

$$\begin{aligned} &\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p \\ &\nabla\cdot\boldsymbol{u} = 0 \end{aligned} \tag{3.1}$$

where ∇p is the pressure force per unit mass and one must specify also the boundary conditions: for simplicity we choose periodic boundary conditions.

The continuity equation is automatically satisfied by introducing the stream function ψ defined as

$$\boldsymbol{u}(\boldsymbol{x}, t) = \nabla^{\perp}\psi(\boldsymbol{x}, t) \tag{3.2}$$

where $\nabla^{\perp} = (-\partial_2, \partial_1)$, and the vorticity ω:

$$\omega = \nabla\times\boldsymbol{u} = \Delta\psi(\boldsymbol{x}, t). \tag{3.3}$$

Then equation (3.1) reduces to

$$\partial_t\omega + (\boldsymbol{u}\cdot\nabla)\omega = 0 \tag{3.4}$$

which expresses the conservation of vorticity in any fluid element. Equations (3.1) cannot be put in Hamiltonian form but Liouville's theorem is still valid (Lee 1952). At variance with the field theories discussed in § 2, an infinite set of integrals of motion exists for (3.1). Therefore many results are known on the solutions of (3.1) (see, e.g., Frisch 1983). Also the direct construction of a statistical mechanics of Euler flows is an open problem. However it is possible to build up a formal equilibrium statistical mechanics of (3.1) in a standard way introducing a truncation (Kraichnan 1975, Salmon *et al* 1976). The ergodic behaviour of the truncated system has been observed in numerical simulations by Kells and Orszag (1978).

Let us perform the truncation by imposing the cutoff $k_{\max}$ on the Fourier series representation of the velocity field

$$\boldsymbol{u}(\boldsymbol{x}, t) = \sum_{|\boldsymbol{k}|<k_{\max}} \hat{\boldsymbol{u}}(\boldsymbol{k}, t)\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}) \tag{3.5}$$

where $\boldsymbol{k}=2\pi\boldsymbol{n}/L$, L being the linear dimension of the system. The continuity equation can be rewritten in the form

$$\boldsymbol{k}\cdot\hat{\boldsymbol{u}}(\boldsymbol{k})=0 \tag{3.6}$$

for every $\boldsymbol{k}$. Moreover the reality condition holds:

$$\hat{\boldsymbol{u}}(-\boldsymbol{k})=\hat{\boldsymbol{u}}^*(\boldsymbol{k}). \tag{3.7}$$

As a consequence $\boldsymbol{u}(\boldsymbol{k})$ can be written in terms of a single scalar real variable $y(\boldsymbol{k})$ and it is easy to express the conservation of the energy, of the vorticity and Liouville's theorem in terms of this new variable as follows:

$$\begin{gathered}\frac{1}{2}\sum_{|\boldsymbol{k}|<k_{\max}} y^2(\boldsymbol{k})=E=\text{constant}\\ \frac{1}{2}\sum_{|\boldsymbol{k}|<k_{\max}} k^2y^2(\boldsymbol{k})=\Omega=\text{constant}\\ \sum_{|\boldsymbol{k}|<k_{\max}}\frac{\partial}{\partial y(\boldsymbol{k})}\left(\frac{\mathrm{d}y(\boldsymbol{k})}{\mathrm{d}t}\right)=0.\end{gathered} \tag{3.8}$$

As for Hamiltonian systems we can define the equilibrium statistical mechanics in the space of $\{y(\boldsymbol{k})\}$ taking into account the two conservation laws (of energy E and enstrophy Ω) and the Liouville's theorem. Hald (1976) has shown that integrals of motion other than enstrophy and energy can be obtained by particular truncations, but Lee (1975a, b, 1977) proved that for non *ad hoc* truncations energy and enstrophy are the only conserved quantities.

The corresponding microcanonical distribution is now

$$P(\{y(\boldsymbol{k})\})\propto\delta\left(\frac{1}{2}\sum_{|\boldsymbol{k}|<k_{\max}}y^2(\boldsymbol{k})-E\right)\delta\left(\frac{1}{2}\sum_{|\boldsymbol{k}|<k_{\max}}k^2y-\Omega\right). \tag{3.9}$$

Introducing the canonical ensemble (see Salmon *et al* 1976) in the usual way, one can obtain, after straightforward calculations,

$$\overline{y^2(\boldsymbol{k})}\sim\overline{|\hat{\boldsymbol{u}}(\boldsymbol{k})|^2}\sim\frac{1}{\beta+\gamma k^2}. \tag{3.10}$$

In the naive continuum limit ($k_{\max}\to\infty$) this formula shows the usual UV catastrophe, consistent with Gibb's formulation of statistical mechanics. Approach to equilibrium (i.e. equation (3.10)) has been observed in many numerical experiments, see for example Basdevant and Sadourny (1975) and Fox and Orszag (1973). Now, let us tackle this problem dynamically. We can express (3.3) in the form

$$\psi(\boldsymbol{x})=\int G(\boldsymbol{x}-\boldsymbol{x}')\omega(\boldsymbol{x}')\,\mathrm{d}^2\boldsymbol{x}' \tag{3.11}$$

where $G(\boldsymbol{x}-\boldsymbol{x}')$ is the Green function associated with the Laplacian operator. As a consequence of (3.2) one can write

$$\boldsymbol{u}(\boldsymbol{x},t)=\int\nabla_{\boldsymbol{x}}^{\perp}G(\boldsymbol{x}-\boldsymbol{x}')\omega(\boldsymbol{x}',t)\,\mathrm{d}^2\boldsymbol{x}'. \tag{3.12}$$

An important consequence of the conservation of vorticity (3.4) is that

$$\max_{\boldsymbol{x}}|\omega(\boldsymbol{x},t)|=\max_{\boldsymbol{x}}|\omega(\boldsymbol{x},0)|. \tag{3.13}$$

Since $|\partial G(\boldsymbol{x})| < \text{constant}/|\boldsymbol{x}|$ (Eidus 1958, Rose and Sulem 1978) from (3.12) and (3.13) it follows after simple calculations (see Sulem and Sulem 1983):

$$|\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x})| < c_1 r |\ln(r/L) - 1| \tag{3.14}$$

where

$$c_1 \sim \max_{\boldsymbol{x}} |\omega(\boldsymbol{x}, 0)| \sim \sqrt{\Omega}. \tag{3.15}$$

From (3.14) with simple manipulations we obtain the bound (see appendix 2)

$$|\hat{\boldsymbol{u}}(\boldsymbol{k})|^2 \leq \text{constant}\, |\boldsymbol{k}|^{-4+\varepsilon} \tag{3.16}$$

for high wavenumber modes.

The conclusion is that a smooth initial condition (i.e. E, $\Omega < \infty$) leads to situations where the high wavenumber modes always contain a poor fraction of the total energy of the system and this statement is in sharp contradiction with the conclusions that one can draw on the basis of a formal equilibrium statistical mechanics, in particular (3.10).

4. Truncation and analytical properties

The results obtained by the dynamical approach in §§ 2 and 3 are in disagreement with the numerical experiments performed on Hamiltonian systems with many degrees of freedom (Livi *et al* 1985) (which can be considered as a discretised version of a classical field theory) and on the Euler equation in two dimensions (Fox and Orszag 1973, Basdevant and Sadourny 1975).

Let us discuss the latter case. In numerical experiments where the initial conditions were chosen far from the 'equipartition' condition (see (3.10)) and such to satisfy the regularity requirements that we have discussed above, the system evolves approaching the thermodynamic equilibrium. However, we will clarify in a moment that the discrepancy with the bound of (3.16) is purely apparent.

In fact one can prove the existence of an analytic continuation of the field $\boldsymbol{u}(\boldsymbol{x}, t)$ as a function of $\boldsymbol{x}$ for any value of t in a finite strip around the real plane $\boldsymbol{x}$ (Sulem and Sulem 1983, Sulem *et al* 1983). This leads to the following estimate for the asymptotic Fourier spectrum:

$$|\hat{\boldsymbol{u}}(\boldsymbol{k})| \sim |\boldsymbol{k}|^{-\alpha} \exp[-\delta(t)|\boldsymbol{k}|] \qquad |\boldsymbol{k}| \gg 1/L$$

where α depends on the nature of the closest singularity to the real plane produced by the analytic continuation of $\boldsymbol{u}(\boldsymbol{x}, t)$, while $\delta(t)$ is proportional to the distance of this singularity from the real plane, i.e. to the width of the analyticity strip (Sulem and Sulem 1983, Sulem *et al* 1983).

Now let us remember that in any numerical simulation of a partial differential equation one must introduce a discretisation of space: the main consequence is the existence of a lower resolution scale in space Δx (which is the length of the mesh or the inverse of the maximal wavenumber $1/k_{\max}$ if one makes the truncation in the Fourier space).

At time t, such that $\delta(t) \gg \Delta x$ everything works well: the numerical simulation gives a proper representation of the continuous system and equipartition of energy is absent. On the contrary, when $\delta(t) < \Delta x$ the discretised system is unable to reproduce the analyticity properties of the field equations. As a consequence equipartition of energy, observed in the numerical experiments, has to be considered as a typical feature of the discretised version of the continuous model.

In two-dimensional fluid dynamics the amplitude of the analyticity strip decays rapidly in time (Sulem *et al* 1983). It does not strongly depend on the details of the initial conditions (Sulem and Sulem 1983); in particular it can depend on $\max_x |\omega(x, 0)|$. A rigorous bound for $\delta(t)$ is given by

$$\ln \delta(t) \geq -c \exp(-\beta t) \qquad c, \beta \sim O(1).$$

However in numerical experiments an exponential decay of $\delta(t)$ is obtained (Sulem *et al* 1983)

$$\delta(t) \sim \exp(-at) \qquad a \sim O(1).$$

Therefore the analyticity properties of the solution of the continuous system are rapidly lost in a numerical experiment. The most careful numerical experiment (256 × 256 square lattice) (see Sulem *et al* 1983) shows an agreement with the dynamical bounds (3.14) and (3.16) until $\delta(t) \geq \Delta x$.

For Hamiltonian systems the behaviour of $\delta(t)$ is much more complicated, because it strongly depends on the details of the initial condition and in particular on the total energy see Fucito *et al* (1982) and Livi *et al* (1983). A hint towards the understanding of this effect is the fact that in the Hamiltonian systems that we have considered there are linear terms which are absent in the Euler equation. However, for large values of the energy strong evidence has been found of a fast relaxation of $\delta(t)$ below the smallest scale of spatial resolution (see Livi *et al* 1983).

Let us point out the two main consequences. The equipartition of energy obtained in a numerical experiment is not necessarily a property of the corresponding continuous system. On the other hand, the absence of equipartition is certainly not an effect of the truncation; high wavenumber modes (i.e. small spatial scales) are very weakly excited in this case. Consequently the continuum limit can be safely performed.

5. Numerical experiment on a model of radiant cavity

As an example, let us analyse the effect of the truncation of a field equation deriving from a simple one-dimensional model of a radiant cavity.

The model was first proposed by Bocchieri *et al* (1972) and subsequently studied numerically by Benettin and Galgani (1982). It describes the electromagnetic field between two fixed infinite parallel mirrors coupled to a charged infinite plane parallel to the mirrors, which is allowed to move in the direction of the mirrors and is acted upon by a non-linear restoring force. The equations of motion are

$$\begin{aligned} &\partial_t^2 A - \partial_x^2 A = 2\sqrt{\pi}\beta \dot{z}\delta(x) \\ &\ddot{z} = -\frac{\beta}{2\sqrt{\pi}}\partial_t A(x=0, t) - \alpha z^3 \\ &A(\pm 1, t) = 0 \end{aligned} \tag{5.1}$$

where $\delta(x)$ is the Dirac delta function.

As these equations are linear in the field variable $A(x, t)$ one can write them down directly in terms of the odd Fourier components of A:

$$\ddot{a}_n + \omega_n^2 a_n = \beta \dot{z} \qquad n = 1, 2, \ldots, \infty$$
$$\ddot{z} = -\beta \sum_{n=1}^{\infty} \dot{a}_n - \alpha z^3 \qquad \omega = \tfrac{1}{2}\pi(z_n - 1). \tag{5.2}$$

In order to perform a numerical integration of the system (5.2) one has to truncate it, limiting the values of n to the interval $[1, N]$. The limit $N \to \infty$ corresponds to the continuum limit of such a model, as defined in the introduction.

It should be observed that the shortest period $T \propto 1/N$ goes to zero as N increases; this fact causes some problems to the integration algorithm (this problem is not present for a model of a solid where a maximal frequency is present). The algorithm is described in appendix 1. We have performed two simulations at a given value of the energy with the same initial condition, but with different values of N: $N = 8$, $N = 16$. In figure 1 we show the value of the energy per mode E_n as a function of n:

$$E_n = \tfrac{1}{2}(\dot{a}_n^2 + \omega_n^2 a_n^2). \tag{5.3}$$

In the first case ($N = 8$) the system has reached equipartition of energy. But approaching the continuum limit ($N = 16$), half of the modes still do not significantly share energy. The energies E_n against n, for $7 \leqslant n \leqslant 16$, lie on an exponential shape as was already observed by Benettin and Galgani (1982). A fit with $\log E_n = c_1 - c_2 n$ gives a correlation coefficient $R \simeq 0.93$. In the second simulation ($N = 16$) the lower modes carry an energy of the same order of the equipartition energy of the system with $N = 8$. Therefore equipartition of energy is an artefact of the truncation of the model and is not a

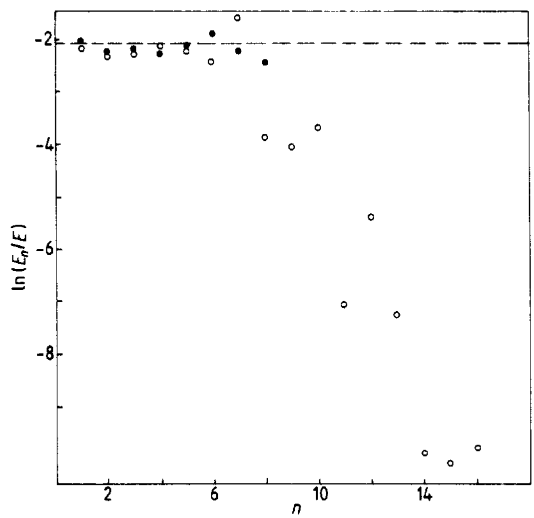

Figure 1. $\text{Ln}(E_n/E)$ plotted against n, where E_n is the harmonic energy per mode averaged over a time interval $T = 10^5$ ($h = 0.01$) and E is the total energy ($E = 1297$), $\beta = 1.18$, $\alpha = 1$. The dots refer to the system with $N = 8$, the circles to $N = 6$. The initial condition is the same in the two cases: only kinetic energy is given to the modes $n = 1$, 2 and to the plate. The broken line indicates equipartition for $N = 8$.

property of the continuum limit. The violation of equipartition is stronger as we get closer to the continuum limit, as was already observed by Patrascioiu *et al* (1985).

If this model could be a realistic representation of the black body we should conclude that the ultraviolet catastrophe is possibly present only when the observation time goes to infinity.

However it is obvious that this model is not a faithful description of the interaction between radiation and matter. For instance, the charged plane behaves dynamically like a mode of the electromagnetic field. Therefore its energy in the continuum limit is zero, while one should expected that the energy carried by matter remains finite. The investigation of a realistic model is an interesting open problem.

6. Conclusions

In this paper we have investigated the possible failure of ergodicity (and hence of ultraviolet catastrophe) in classical field theories. In some cases (certain Lagrangian fields and two-dimensional Euler flow) it is possible to show that the UV catastrophe is incompatible with the dynamics if the initial conditions are smooth. The same property (i.e. the absence of the UV catastrophe) is a general behaviour of all classical field equations of motion whose solutions have analyticity properties.

Moreover, by comparing analyticity properties and numerical simulations of two-dimensional Euler equations and of a simple one-dimensional field theory, we have shown that it is very easy to arrive at the wrong conclusions from the numerical results on discrete approximations of continuous fields. Indeed also for systems with good analyticity properties (i.e. without the UV catastrophe) the truncated equations (i.e. an approximation with a finite number of degrees of freedom) can show an approach to equipartition. This apparent paradox is due to the fact that in a finite time the truncated equations lose the analyticity properties of the continuous systems. Therefore ergodicity is a property of the finite system but not of the field theory. Agreement between truncated systems and the corresponding field theory increases, of course, with the number of degrees of freedom taken into account in the truncated system.

However this is true only at finite times and it often happens that, also with a very large finite system, after a 'short' time there is not enough accuracy to resolve the analyticity properties of the continuous system.

Appendix 1

In this appendix we describe the numerical algorithm used in the integration of (5.2). We have developed this algorithm because the standard methods (Euler-Cauchy or Runge-Kutta) do not give a good conservation of the total energy. Moreover it is not possible to use the leapfrog method because of the terms $\dot{a}_n$ in (5.2). Our numerical procedure is slightly different from that used in Benettin and Galgani (1982). Indeed we work directly with (5.2).

Let us fix a finite time step h and consider the Taylor series expansion of a generic function $x(t)$:

$$x(t+h)=x(t)+\dot{x}(t)h+\tfrac{1}{2}\ddot{x}(t)h^2+\tfrac{1}{6}\dddot{x}(t)h^3+O(h^4) \quad \text{(A1.1}a\text{)}$$

$$x(t-h)=x(t)-\dot{x}(t)h+\tfrac{1}{2}\ddot{x}(t)h^2-\tfrac{1}{6}\dddot{x}(t)h^3+O(h^4). \quad \text{(A1.1}b\text{)}$$

From these relations one obtains

$$\ddot{x}(t)=(1/h^2)[x(t+h)-2x(t)+x(t-h)]+O(h^2) \tag{A1.2a}$$

$$\dot{x}(t)=(1/2h)[x(t+h)-x(t-h)]+O(h^2). \tag{A1.2b}$$

We want to point out that this definition of velocity $\dot{x}$ reduces the error to the same order as in the determination of the acceleration $\ddot{x}$.

Applying this scheme to the variables $x(t)$ and $z(t)$ defined in § 5, (5.2) reduce to

$$\frac{1}{h^2}[z(t+h)-2z(t)+z(t-h)]=-\beta\sum_{n=1}^{N}\frac{1}{2h}[a_n(t+h)-a_n(t-h)]-\alpha z^3(t) \tag{A1.3a}$$

$$\frac{1}{h^2}[a_n(t+h)-2a_n(t)+a_n(t-h)]=\beta\frac{1}{2h}[z(t+h)-z(t-h)]-\omega_n^2a_n(t). \tag{A1.3b}$$

Now we introduce the new variables

$$w(t)=\frac{z(t+h)-z(t)}{h} \tag{A1.4a}$$

$$b_n(t)=\frac{a_n(t+h)-a_n(t)}{h}. \tag{A1.4b}$$

Then, (4.3) can be rewritten in terms of the new variables as follows:

$$w(t+h)=w(t)-h\left(\sum_{n=1}^{N}\beta\frac{b_n(t+h)+b_n(t)}{2}+\alpha z^3(t+h)\right) \tag{A1.5a}$$

$$b_n(t+h)=b_n(t)+h\left(\beta\frac{w(t+h)+w(t)}{2}-\omega_n^2a_n(t+h)\right) \tag{A1.5b}$$

Substituting (A1.5*b*) in the RHS of (A1.5*a*) one obtains

$$\chi_+w(t+h)=\chi_-w(t)-\beta h\sum_{n=1}^{N}b_n(t)+\tfrac{1}{2}\beta h^2\sum_{n=1}^{N}\omega_n^2a_n(t+h)-\alpha hz^3(t+h) \tag{A1.6a}$$

where $\chi_\pm=1\pm\frac{1}{4}\beta^2h^2N$.

Substituting (A1.6*a*) in the RHS of (A1.5*b*) one obtains the relation:

$$b_n(t+h)=b_n(t)+\tfrac{1}{2}\beta h(1+\chi_-/\chi_+)w(t)-h\omega_n^2a_n(t+h)-\tfrac{1}{2}\alpha\beta h^2z^3(t+h)$$
$$-\tfrac{1}{2}\beta^2h^2\sum_{m=1}^{N}b_m(t)+\tfrac{1}{4}\beta^2h^3\sum_{m=1}^{N}\omega_m^2a_m(t+h). \tag{A1.6b}$$

Consequently, the numerical algorithm is defined by the system formed by (A1.4) and (A1.6).

Let us observe that the beginning of the integration procedure one has to fix the initial conditions for $z(t)$, $a_n(t)$, $w(t)$ and $b_n(t)$; then (A1.4) give $z(t+h)$ and $a_n(t+h)$, which are then substituted in (A1.6) to give $w(t+h)$ and $b_n(t+h)$, and so on. The 'velocities' of the system are expressed by the formulae

$$\dot{a}_n(t+h)=\tfrac{1}{2}(b_n(t+h)+b_n(t))+O(h^2) \tag{A1.7a}$$

$$\dot{z}(t+h)=\tfrac{1}{2}(w(t+h)+w(t))+O(h^2). \tag{A1.7b}$$

Appendix 2

In this appendix we sketch the derivation of (3.16) from (3.14). If

$$u(x)=\sum_k \hat{u}(k)\exp(ik\cdot x) \tag{A2.1}$$

then

$$\langle|u(x+r)-u(x)|^2\rangle=2\sum_k(1-\cos k\cdot r)|\hat{u}(k)|^2<\text{constant } r^{2-\varepsilon} \tag{A2.2}$$

where ⟨ ⟩ stands for space average. Noting that

$$\sum_k(1-\cos k\cdot r)|\hat{u}(k)|^2\geqslant\sum_{\frac{1}{2}\tilde{k}<k<\tilde{k}}(1-\cos k\cdot r)|\hat{u}(k)|^2 \tag{A2.3}$$

and that for $\tilde{k}\cdot r\leqslant 1$ it is $1-\cos\tilde{k}\cdot r\sim\frac{1}{2}(\tilde{k}\cdot r)^2$, one obtains

$$\sum_k(1-\cos k\cdot r)|\hat{u}(k)|^2\geqslant\text{constant}(\tilde{k}\cdot r)^2|\hat{u}(k)|^2|\tilde{k}|^2. \tag{A2.4}$$

As a consequence we find

$$\tilde{k}^4r^2|\hat{u}(k)|^2\leqslant\text{constant}\times r^{2-\varepsilon}. \tag{A2.5}$$

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